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On Bilinear Invariant Differential Operators Acting on Tensor Fields on the Symplectic Manifold

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Abstract

Let M be an n-dimensional manifold, V the space of a representation $\rho : GL(n) \longrightarrow GL(V)$. Locally, let $T(V)$ be the space of sections of the tensor bundle with fiber V over a sufficiently small open set $U \subset M$, in other words, $T(V)$ is the space of tensor fields of type V on M on which the group $\mathrm{Diff}(M)$ of diffeomorphisms of M naturally acts. Elsewhere, the author classified the $\mathrm{Diff}(M)$-invariant differential operators $D : T(V_1) \otimes T(V_2) \longrightarrow T(V_3)$ for irreducible fibers with lowest weight. Here the result is generalized to bilinear operators invariant with respect to the group $\mathrm{Diff}_\omega(M)$ of symplectomorphisms of the symplectic manifold (M, ω). We classify all first order invariant operators; the list of other operators is conjectural. Among the new operators we mention a 2nd order one which determins an "algebra" structure on the space of metrics (symmetric forms) on M.

Let ρ be a representation of the group $Sp(2m; \mathbb{R})$ in a V_ρ. A *tensor field of type* ρ on a $2m$-dimensional symplectic manifold M is an object t defined in each local coordinate system x, in which the symplectic form is of the canonical form $\omega = \sum\limits_{i \le m} dx_i \wedge dx_{2m+1-i}$, by the vector $t(x) \in V_\rho$, where the collections of all vectors $t(x)$ are such that the passage to other coordinates, y (with the same property), is defined by the formula

$$t(y(x)) = \rho\left(\frac{\partial y(x)}{\partial x}\right) t(x).$$

Traditionally (see reviews [4, 5]) the fibers of the tensor bundles were considered finite dimensional, but Leites showed recently [7] that on supermanifolds it is natural and fruitful to consider infinite dimensional fibers: this leads to semi-infinite cohomology of supermanifolds. Similar problem for symplectic manifolds and supermanifolds was not studied yet.

The space of smooth tensor fields of type ρ will be denoted by $T(\rho)$ or by $T(\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is the lowest (or, for finite dimensional representations, highest, for convenience) weight of the irreducible representation ρ.

In what follows the letters ρ, σ, τ will denote irreducible representations of $Sp(2m;\mathbb{R})$ and letters λ, μ, ν their highest weights. (We should have concidered lowest weight only, but in this report we stick to finite dimensional representations, where it does not matter.)

Examples of spaces of tensor fields:

a) $T(0) = C^\infty(M)$;

b) $T(1,0,\dots,0) \cong \mathrm{Vect} \cong \Omega^1$ is the space of vector fields or (which is the same on any symplectic manifold thanks to the nondegeneracy of ω) the space of 1-forms on M;

c) $\prod^r = T(\underbrace{1,\dots,1}_{r\text{-many}},0,\dots,0)$ the space of primitive r-forms.

Remark 1. Observe that the spaces of tensor fields traditionally are understood as p times covariant and q times contravariant ones (T^p_q) or their subspaces subject to some symmetry conditions. Such tensors split into the direct sum of irreducible $Sp(2n)$-modules and the same module can be encountered in distinct T^p_q (with different p's and q's). For example, the tensors of trivial type, $T(0)$ can be encountered in $T^0_0 = C^\infty(M)$, as stated above, and also in $T^2_0 = C^\infty(M)\omega$ and in many other places. In this sense the case of symplectomorphisms differs from the general diffeomorphisms, where each irreducible $GL(n)$-module has a unique embedding into the tensor algebra.

A differential operator $B : T(\rho_1)\otimes T(\rho_2)\otimes\cdots\otimes T(\rho_n) \longrightarrow T(\tau)$ is called *n-ary* (*un*ary, *bi*nary, etc. for $n = 1, 2$, respectively). Such an operator B is called $\mathrm{Diff}(M)$-*invariant* if it is uniquely expressed in all coordinate systems; it is $\mathrm{Diff}_\omega(M)$-*invariant* if it is uniquely expressed in all coordinate systems in which the symplectic form is of the standard (canonical) form.

The unary $\mathrm{Diff}_\omega(M)$-invariant differential operators:

A N Rudakov classified all such operators ([8, 9]):

0-th order: the multiplication by a scalar;

1-st order: the derivations of the primitive forms $d_+ : \prod^r \longrightarrow \prod^{r+1}$ and $d_- : \prod^{r+1} \longrightarrow \prod^r$ $(0 \le r \le m-1)$. These operators are compositions of the exterior differential $d : \Omega^p \longrightarrow \Omega^{p+1}$ and the projection onto the space of primitive forms; recall that $\Omega^p = \prod^p \oplus \prod^{p-2}\omega \oplus \prod^{p-4}\omega^2 \oplus \dots$ for $p \le m$ and $\Omega^p \cong \Omega^{2m-p}$;

2-nd order: $d_2 = d_+ \circ d_- : \prod^r \longrightarrow \prod^r$ $(1 \le r \le m)$.

Remark 2. Rudakov's theorem implies that other invariant operators that might spring to mind ($d_- \circ \omega \circ d_-$, etc.) are multiples of the described ones.

The binary $\mathrm{Diff}_\omega(M)$-invariant differential operators:

On the space $T_c(\rho)$ of tensor fields of type ρ with compact support, as indicated by the subscript, there is an invariant inner product

$$\langle \chi, \theta \rangle = \int_M \langle \chi(x), \theta(x) \rangle \omega_0^m, \tag{IP}$$

where $\langle\cdot,\cdot\rangle$ in the integrand is the $Sp(2m;\mathbb{R})$-invariant inner product on V_ρ. Strictly speaking, this duality has no analog for tensor fields with formal coefficients but we use it to formally extend the notion in order to define the following 1-*dual* and 2-*dual* spaces of the space $T(\rho_1)\otimes T(\rho_2)$, as the duality with respect to the first (or second) factor.

Clearly, if $B : T(\rho_1) \otimes T(\rho_2) \longrightarrow T(\tau)$ is a $\mathbf{Diff}_{\boldsymbol{\omega}}\boldsymbol{(M)}$-invariant differential operator, then the operators $B^{*1} : T(\tau) \times T(\rho_2) \longrightarrow T(\rho_1)$ and $B^{*2} : T(\rho_1) \times T(\tau) \longrightarrow T(\rho_2)$, the 1-dual and 2-dual of B with respect to the inner product (IP), are also differential and invariant ones.

The isomorphisms between various realizations of $T(\rho)$ spoken about in Remark 1 are, clearly, 0-th order invariant differential operators. Our description of invariant operators is given up to such isomorphisms. For the classification of binary operators invariant with respect to the group of general diffeomorphisms see [1], for preliminary results on $\mathrm{Diff}_\omega(M)$-invariant differential operators see [2]. The results of this paper were preprinted in [3].

0-th order operators are of the form

$$Z(\chi, \theta) = \mathrm{pr}(\chi(x) \otimes \theta(x)),$$

where $\mathrm{pr} : V_\rho \otimes V_\sigma \longrightarrow V_\tau$ is the projection of the tensor product onto an irreducible component.

1-st order operators are given by the following theorem

Theorem. *Any bilinear* 1*-st order (with respect to all arguments)* $\mathrm{Diff}_\omega(M)$*-invariant differential operator* $B : T(\lambda) \otimes T(\mu) \longrightarrow T(\nu)$ *is a linear combination of the following cases* P1–P8 *(some of which host several distinct operators being restricted onto tensors with irreducible fibers) and the operators obtained from them by* 1*-dualization or* 2*-dualization or transposition of the arguments.*

P1) $\lambda = (\underbrace{1, \dots, 1}_{p\text{-many } 1\text{'s}}, 0, \dots, 0)$; weights μ and ν differ from each other by a unit in r places, $r \equiv p + 1 \mod 2$. For $r \leq p + 1$ there exists a representation of these operators in the form $Z(d_+\omega, \theta)$ and for $r \leq p - 1$ there exists a representation of these operators in the form $Z(d_-\omega, \theta)$.

P2) The Lie derivative being restricted onto $Sp(2m; \mathbb{R})$-irreducible subspaces splits into several operators of the form $Z(d_\pm\omega, \theta)$ and an operator

$$L : \mathrm{Vect} \times T(\rho) \longrightarrow T(\rho)$$

which cannot be reduced to operators of the form P1).

Remark 3. Observe that if $\xi \in \mathfrak{h}(M) \subset \mathfrak{vect}(M)$ is a Hamiltonian vector field, then, by identifying $\mathfrak{h}(M)$ with $d\Omega^0$, we see that $d_+\xi = d_-\xi = 0$ and in this case L coincides with the Lie derivative. Therefore, L determines a representation of the Lie algebra $\mathfrak{h}(M)$ in the space $T(\rho)$. It is not difficult to show that the invariance of B is equivalent to its $\mathfrak{h}(M)$-invariance:

$$L(\xi, B(\chi, \theta)) = B(L(\xi, \chi), \theta) + B(\chi, L(\xi, \theta))$$

for any $\chi \in T(\rho)$, $\theta \in T(\sigma)$, $\xi \in \mathfrak{h}(M)$.

P3) $S^k\mathfrak{vect}\times S^l\mathfrak{vect} \longrightarrow S^{k+l-1}\mathfrak{vect}$ (clearly, $S^k\mathfrak{vect} \cong T(k,0,\dots,0)$) is the Poisson bracket (a.k.a. the *symmetric Schouten's concomitant*) on (polynomial in momenta) functions on T^*M.

P4) λ, μ, ν are vectors of the form $(2,1,\dots,1,0,\dots,0)$ each, with p, q and r non-zero coordinates, respectively, such that $p+q+r \equiv 0 \mod 2$, $|p-q| \leq r \leq p+q$, and $p+q+r \leq 2m+2$.

If all inequalities are strict, then there exist *four distinct operators* defined on the spaces of such fields, otherwise there exist only *two* distinct operators. For $p+q+r \leq 2m$ two of these four or two operators are obtained as restrictions of the *Nijenhuis bracket*, or its conjugates, onto the subspaces

$$T(2,1,\dots,1,0,\dots,0) \subset \Omega^p \otimes_{C^\infty(M)} \mathfrak{vect}.$$

Remark 4. The remaining two operators (i.e., the ones which are not the restrictions of the Nijenhuis bracket) are new. I do not know anything about them except that they exist and the same applies to the following two cases P5) and P6).

P5) λ, μ are of the same form as for P4), $\nu = (3,1,1,\dots,1,0,\dots,0)$. There exists one operator for $|p-q|+1 \leq r \leq p+q-1$, $p+q+r \equiv 1 \mod 2$, $p+q+r \leq 2m+1$.

P6) λ, μ are the same as in 4), $\nu = (2,2,1,\dots,1,0,\dots,0)$ with r non-zero entries. The operator exists under the same conditions on p, q, r as for P5).

P7) $\nu = (1,\dots,1,0,\dots,0)$; whereas λ, μ and conditions on p, q, r are the same as in 5). In this case there exists a unique operator which is not reducible to operators of the form $d_\pm Z$. It is a restriction of the Nijenhuis bracket.

P8) $\lambda = (2,0,\dots,0)$; whereas μ and ν differ from each other by a unit at one place. There exists a unique such operator. Further on I will give arguments which enable one to express it, in principle, explicitly.

2nd order operators:

I could not *classify* such operators so far. However, I was lucky to find one new invariant operator, denoted in the literature Gz:

$$Gz : T(2,0,\dots,0) \times T(2,0,\dots,0) \longrightarrow T(2,0,\dots,0).$$

For $m=1$ I got the explicit expression for the operator Gz in 1976. Let me reproduce it. In coordinates x, y we have $\omega = dx \wedge dy$. Then

$$\begin{aligned}
Gz : {}& a\cdot dx^2 + 2b\cdot dxdy + c\cdot dy^2, a'\cdot dx^2 + 2b'\cdot dxdy + c'\cdot dy^2 \\
&\mapsto \frac{\partial^2 g}{\partial x^2}dx^2 + 2\frac{\partial^2 g}{\partial x\partial y}dxdy + \frac{\partial^2 g}{\partial y^2}dy^2 + (\{c,a'\} - \{a,c'\})dxdy \\
&+ \left(\frac{\partial^2 a}{\partial y^2} - 2\frac{\partial^2 b}{\partial x\partial y} + \frac{\partial^2 c}{\partial x^2}\right)\left(a'dx^2 + 2bdxdy + c'dy^2\right) + \left(\{a,b'\} - \{b,a'\}\right)dx^2 \\
&+ \left(\frac{\partial^2 a}{\partial y^2} - 2\frac{\partial^2 b}{\partial x\partial y} + \frac{\partial^2 c}{\partial x^2}\right)\left(adx^2 + 2bdxdy + cdy^2\right)\left(\{b,c'\} - \{c,b'\}\right)dy^2,
\end{aligned}$$

where $g = ac' - 2bb' + ca'$ and $\{\cdot,\cdot\}$ is the Poisson bracket. An explicit form of Gz for $m > 1$ is to be found.

Sketch of the proof of the Theorem:

Set $y_i = x_{2m+1-i}$ $(1 \le i \le m)$, $\partial_i = \frac{\partial}{\partial x_i}$, $\delta_i = \frac{\partial}{\partial y_i}$. Denote the elements of the Lie algebra $\mathfrak{sp}(2m;\mathbb{R}) \subset \mathfrak{h}(M)$ by

$$e^{ii} = y_i\partial_i, \qquad e_{ii} = x_i\delta_i, \qquad e^i_j = x_j\partial_i + y_i\delta_j.$$

Then, clearly,

$$e^{ij} = e^{ji} = y_i\partial_j + y_j\partial_i \qquad \text{and} \qquad e_{ij} = e_{ji} = x_i\delta_j + x_j\delta_i \qquad \text{for} \quad i \ne j.$$

Let $I(\rho)$ be the space of differential operators from $T(\rho)$ into $C^\infty(M)$ with constant coefficients, i.e.,

$$I(\rho) = \left\{\sum P_i(\partial,\delta)u_i \mid u_i \in V^*_\rho \cong V_\rho\right\}.$$

The grading in $I(\rho)$ is induced by that in the space of polynomials P_i's, i.e., $I(\rho)_0 \cong V_\rho$. Define the pairing $I(\rho) \times T(\rho) \longrightarrow \mathbb{R}$ by the formula

$$\langle Pu, x\rangle = P(\langle u, \chi(x)\rangle)|_{x=0}.$$

On $I(\rho)$, define the $\mathfrak{h}(M)$-action, dual to the action on $T(\rho)$, via L. Now, to describe the invariant operators it suffices to find all the $\mathfrak{h}(M)$-morphisms $I(\tau) \longrightarrow I(\rho) \otimes_{\mathbb{R}} I(\sigma)$. It turns out that such a morphism is completely defined by the image of the highest vector $v \in V_\tau = I(\tau)_0$. Here we have fixed a Borel subalgebra $\left\{\sum\limits_{i\le j} a_{ij}x_j\partial_i\right\} \bigcap \mathfrak{sp}(2m;\mathbb{R})$ so that $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ can be the image of a highest weight singular vector if and only if

$$e^i_{i+1}w = 0 \quad \text{for} \quad 1 \le i \le m-1 \qquad \text{and}$$
$$e^{m,m}w = 0 \qquad \text{(conditions on } w \text{ to be a highest vector)}$$

and

$$\left(x_1^2\delta_1\right) w = 0 \qquad \text{(conditions of } \textit{singularity} \text{ of the vector)}$$

The degree of $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ is equal to the order of the corresponding differential operator. The general form of a vector of degree 1 is

$$w = \sum_{i\le m} \partial'_i z^0_i + \delta'_i t^0_i + \partial''_i z^1_i + \delta''_i t^1_i,$$

where $z^j_i, t^j_i \in V_\rho \otimes V_\sigma$, $\partial'(u\otimes v) = \partial u \otimes v$, $\partial''(u \otimes v) = u \otimes \partial v$. If w is a highest vector, then all vectors z, t are expressed in terms of z^0_1, z^1_1 which should satisfy

$$e^i_{i+1}z^j_1 = 0 \qquad \text{for} \quad 2 \le i \le m-1, \qquad \left(e^1_2\right)^2 z^j_1 = 0, \qquad e^{m,m}z^j_1 = 0.$$

The condition $\left(x_1^2\delta_1\right) w = 0$ is equivalent to the equation

$$e'_{1,1}z^0_1 + e''_{1,1}z^1_1 = 0,$$

where (double) prime means that the operator acts only on the first (second) multiple of the tensor product.

I have succeeded to define all the cases, where the above system possesses a solution in $V_\rho \otimes V_\sigma$; though in certain cases I was not able to find the solution itself.

Here is an example of a successfully solved case (case 8)):

$$\lambda = (2, 0, \dots, 0), \qquad \nu = (\mu_1, \dots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \dots);$$

the case $\nu_k = \mu_k - 1$ is dual to this one. Let $u_0 \in V_\rho$ be a highest vector, then

$$u_0 \otimes v - \frac{1}{2} \sum_{2 \le i \le k} e_1^i u_0 \otimes e_i^1 v \in V_\rho \otimes V_\sigma$$

is a highest vector of weight $(\nu_1 + 1, \nu_2, \dots, \nu_m)$. We conclude that

$$\begin{aligned} z_1^0 = u_0 \oplus e_{11} v &- \sum_{2 \le i \le k} e_1^i u_0 \otimes e_{11} e_i^1 v - \frac{1}{2} \sum_{2 \le i < j \le m} e_1^i e_1^j u_0 \otimes \left(e_{ij} + e_{1j} e_1^i + e_{1i} e_j^1\right) v \\ &- \frac{1}{2} \sum_{2 \le i \le k} (e_1^i)^2 u_0 \otimes \left(e_{ii} + e_{1i} e_i^1\right) v \\ &+ \frac{1}{2} \sum_{2 \le i \ne j \le m} e_1^i e_{1j} u_0 \otimes \left(e_i^j + e_1^j e_i^1\right) v + \left(\nu_i + e_1^i e_i^1\right) v \\ &+ \frac{1}{4} \sum_{2 \le i \le m} e_{ii} \left(e_1^i\right)^2 u_0 \otimes + \frac{\nu_1 - 1}{2} \sum_{2 \le i \le k} e_{11} e_1^i u_0 \otimes e_i^1 v, \end{aligned}$$

and

$$z_1^1 = -e_{11} u_0 \otimes v + \sum_{2 \le i \le k} e_{11} e_1^i u_0 \otimes e_i^1 v.$$

Conjectures:

1) *The operator Gz is a particular case of a more general operator*:

$$Gz_{r,s} : T(2, \underbrace{1, \dots, 1}_{r\text{-many}}, 0, \dots, 0) \times T(2, \underbrace{1, \dots, 1}_{s\text{-many}}, 0, \dots, 0) \longrightarrow T(2, \underbrace{1, \dots, 1}_{(r+s)\text{-many}}, 0, \dots, 0).$$

2) *Operators of order > 2 are compositions of operators of orders ≤ 2. There are no operators of order > 5.*

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